Rayleigh-Taylor Instability of Newtonian and Oldroydian Viscoelastic Fluids in Porous Medium

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The Rayleigh-Taylor instability of viscous and viscoelastic (Oldroydian) fluids, separately, has been considered in porous medium. Two uniform fluids separated by a horizontal boundary and the case of exponentially varying density have been considered in both viscous and viscoelastic fluids. The effective interfacial tension succeeds in stabilizing perturbations of certain wave numbers (small wavelength perturbations) which were unstable in the absence of effective interfacial tension, for unstable configuration/stratification.

1. Introduction

A detailed account of the Rayleigh-Taylor instability of Newtonian viscous fluids has been given by Chandrasekhar [1]. Oldroyd [2] proposed a theoretical model for a class of viscoelastic fluids. An experimental demonstration by Toms and Strawbridge [3] has revealed that a dilute solution of methyl methacrylate in *n*-butyl acetate agrees well with the theoretical model of Oldroyd. The medium has been considered to be non-porous in all these studies.

A theoretical and experimental investigation of the instability of slow, immiscible, viscous liquid-liquid displacement in porous media has been made by Chuoke et al. [4]. In flows through porous media, the front is not sharp (as in ordinary fluid dynamics) but is dispersed and broad. Chuoke et al. [4] have assumed a macroscopic interface and an 'effective interfacial tension'.

Considering a viscoelastic fluid, e.g. a dilute solution of methyl methacrylate in *n*-butyl acetate, and an ordinary viscous fluid, the present paper deals with the Rayleigh-Taylor instability of Newtonian and Oldroydian viscoelastic fluids in porous media. The problem finds its usefulness in chemical technology and geophysical fluid dynamics.

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2. Newtonian Fluid. Perturbation Equations

The initial stationary state, whose stability we wish to examine, is that of an incompressible fluid of variable density and viscosity arranged in horizontal strata in a porous medium of variable porosity and permeability. The character of the equilibrium of this initial static state is determined by supposing that the system is slightly disturbed and then following its further evolution.

In flows through porous media there are no sharp fronts and so no actual interfacial tensions at some prescribed levels z_s, as in ordinary fluid dynamics. However, there is a macroscopic interface (broad front) if viewed from a large distance, and by analogy with Laplace's formula, at each point of the macroscopic interface

$$(p_1 - p_2)_{z=z_s} = -T_s(c_1 + c_2), \tag{1}$$

where c_1, c_2 are the signed principal curvatures of the macroscopic interface and T_s is the 'effective interfacial tension'. This is the first approximation to the problem since in practice there is no 'effective surface tension' but in the absence of any better theory, this is being used. This theory was enunciated by Chuoke et al. [4].

Let $\delta\varrho$, δp and $\boldsymbol{q}(u, v, w)$ denote respectively the perturbations of the density ϱ, pressure p and velocity $\boldsymbol{v}(0, 0, 0)$. Let the discontinuity in density occur at $z = z_s$, which after perturbation becomes

$$z_s + \delta z_s(x, y, t). \tag{2}$$

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Therefore, on account of (1), the discontinuity in the normal stresses required for equilibrium is

$$T_s\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta z_s. \tag{3}$$

Hence, the linearized perturbed equations of motion and continuity in porous medium are

$$\frac{\varrho}{\varepsilon}\frac{\partial \boldsymbol{q}}{\partial t}=-\nabla\delta p-g\,\delta\varrho\,\boldsymbol{\lambda} -\frac{\mu}{k}\boldsymbol{q}+\boldsymbol{n}_s\left[T_s\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta z_s\right]\delta(z-z_s), \tag{4}$$

$$\nabla\cdot\boldsymbol{q}=0, \tag{5}$$

where $\boldsymbol{n}_s$ denotes the normal to the macroscopic interface. ε, k, and μ stand for medium porosity, medium permeability and fluid viscosity respectively. $\boldsymbol{\lambda}=(0,0,1)$.

Since the density of a particle moving with the fluid remains unchanged, we have

$$\varepsilon\frac{\partial\varrho}{\partial t}+(\boldsymbol{\varrho}\cdot\nabla)\,\varrho=0,$$

which after perturbation, using initial conditions and on linearization, yields

$$\varepsilon\frac{\partial}{\partial t}\delta\varrho=-w\frac{\mathrm{d}\varrho}{\mathrm{d}z}. \tag{6}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y, and t is given by

$$\exp(i\,\alpha_x x+i\,\alpha_y y+n\,t), \tag{7}$$

where α_x, α_y are horizontal wave number, $\alpha=(\alpha_x^2+\alpha_y^2)^{1/2}$ and n is a complex constant.

Equations (4)–(6), using expression (7), give

$$\frac{\varrho}{\varepsilon}n\,u=-i\,\alpha_x\,\delta p-\frac{\mu}{k}u, \tag{8}$$

$$\frac{\varrho}{\varepsilon}n\,v=-i\,\alpha_y\,\delta p-\frac{\mu}{k}v, \tag{9}$$

$$\frac{\varrho}{\varepsilon}n\,w=-D\,\delta p-\frac{\mu}{k}w -g\,\delta\varrho+T_s\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta z_s\right]\delta(z-z_s), \tag{10}$$

$$i\,\alpha_x u+i\,\alpha_y v+D\,w=0, \tag{11}$$

$$\varepsilon\,n\,\delta\varrho=-w\,D\,\varrho, \tag{12}$$

where $D=\dfrac{\mathrm{d}}{\mathrm{d}z}$. In (10), δz_s can be expressesd in terms of the normal component of the velocity w_s at z_s since

$$\varepsilon\frac{\partial}{\partial t}(\delta z_s)=w_s,\quad \text{i.e. } \delta z_s=\frac{w_s}{\varepsilon\,n}, \tag{13}$$

where the subscript "s" indicates the value of the quantity at $z=z_s$.

Eliminating u, v, and δp between (8)–(10) and using (11) and (12), we obtain

$$D\left[\left(\frac{\varrho}{\varepsilon}+\frac{\mu}{nk}\right)D\,w\right]=\alpha^2\left[\left(\frac{\varrho}{\varepsilon}+\frac{\mu}{nk}\right)D\,w -\frac{g\,(D\,\varrho)\,w}{\varepsilon\,n^2}+\frac{\alpha^2\,T_s w_s}{\varepsilon\,n^2}\delta(z-z_s)\right]. \tag{14}$$

a) Two Uniform Fluids Separated by a Horizontal Boundary

Let two uniform fluids of densities and viscosities ϱ_1, μ_1; ϱ_2, μ_2, and medium porosites and permeabilities ε_1, k_1; ε_2, k_2 be separated by a horizontal boundary at $z=0$. Then for both regions of the fluid, (14) reduces to

$$(D^2-\alpha^2)\,w=0. \tag{15}$$

Since w must vanish both when

$z\to-\infty$ (in the lower fluid)

and

$z\to+\infty$ (in the upper fluid),

and since w must be continuous across the interface at $z=0$, we must suppose that

$$w_1=A\,e^{+\alpha z},\quad (z<0) \qquad w_2=A\,e^{-\alpha z},\quad (z>0), \tag{16}$$

where A is an arbitrary constant.

Another boundary condition, which in view of (14) must be satisfied at an interface between two fluids is

$$\Delta_0\left[\left(\varrho_0+\frac{\mu\,e}{\mu\,k}\right)D\,w\right]=-\frac{\alpha^2}{n^2}\left[g\,\Delta_0(\varrho)-\alpha^2\,T\right]w_0, \tag{17}$$

where w_0 is the value of w at $z=0$ and $\Delta_0(f)$ is the jump which a quantity f experiences at the interface $z=0$. Applying the condition (17) to the solutions (16), we obtain

$$(\varrho_2+\varrho_1)\,n^2+\left(\frac{\mu_2\,\varepsilon_2}{k_2}+\frac{\mu_1\,\varepsilon_1}{k_1}\right)n -\alpha\left[g(\varrho_2-\varrho_1)-\alpha^2\,T\right]=0. \tag{18}$$

It is clear from (18) that the system is stable if $\varrho_2<\varrho_1$ and is unstable for $\varrho_2>\varrho_1$ for all wave numbers in the range $0<\alpha<\alpha_c$, where

$$\alpha_c=\left[\frac{g(\varrho_2-\varrho_1)}{T}\right]^{1/2}. \tag{19}$$

But for all perturbations with wave numbers $\alpha>\alpha_c$ the system is stable.

The surface tension, therefore, succeeds in stabilizing a potentially unstable arrangement ($\varrho_2>\varrho_1$) for all sufficiently short wavelengths but is unable to stabilize all sufficiently long wavelengths which remain unstable.

b) The Case of Exponentially Varying Stratifications

Assume stratifications in fluid density, fluid viscosity, medium porosity, medium permeability and surface tension of the form

$$\begin{aligned}&\varrho=\varrho_0\,e^{\beta z},\quad \mu=\mu_0\,e^{\beta z},\quad \varepsilon=\varepsilon_0\,e^{\beta z},\\ &k=k_0\,e^{\beta z},\quad T_s=T_{s0}\,e^{\beta z},\end{aligned} \tag{20}$$

where ϱ_0, μ_0, ε_0, k_0, T_{s0}, and β are constants. Stratifications of the form (20) are used in order to find a simple analytical solution. Using stratifications of the form (20), (13) transforms to

$$\left(\frac{\varrho_0}{\varepsilon_0}+\frac{\mu_0}{n\,k_0}\right)(D^2-\alpha^2)\,w=\frac{\alpha^2}{\varepsilon_0\,n^2}(\alpha^2\,T_{s0}-g\,\varrho_0\,\beta)\,w. \tag{21}$$

The general solution of (20) is

$$w=A\,e^{m_1 z}+B\,e^{m_2 z}, \tag{22}$$

where A, B are two arbitrary constants, m_1, m_2 are given by the equation

$$m_{1,2}=\pm\alpha\left[1+\frac{M}{L}\right]^{1/2}, \tag{23}$$

and

$$M=\frac{1}{\varepsilon_0\,n^2}(\alpha^2\,T_{s0}-g\,\varrho_0\,\beta),\quad L=\left(\frac{\varrho_0}{\varepsilon_0}+\frac{\mu_0}{n\,k_0}\right). \tag{24}$$

If the fluid is supposed to be confined between two rigid planes at $z=0$ and $z=d$, then the vanishing of w at $z=0$ is given by

$$w=A\,(e^{m_1 z}-e^{m_2 z}), \tag{25}$$

and the vanishing of w at $z=d$ requires

$$e^{(m_1-m_2)d}=1=e^{2is\pi}, \tag{26}$$

which implies that

$$(m_1-m_2)\,d=2\,i\,s\,\pi, \tag{27}$$

where s is an integer.

Inserting the values of m_1, m_2 from (23) in (27), we obtain

$$1+\frac{M}{L}=-\frac{s^2\,\pi^2}{\alpha^2\,d^2},$$

which on simplification gives the dispersion relation

$$n^2+\frac{\varepsilon_0\,\nu_0}{k_0}\,n+\left(\frac{\alpha^2\,d^2}{\alpha^2\,d^2+s^2\,\pi^2}\right)\left(\frac{\alpha^2\,T_{s0}}{\varrho_0}-g\,\beta\right)=0. \tag{28}$$

If $\beta<0$ (stable stratifications), (28) does not allow any positive root of n, and so the system is always stable for disturbances of all wave numbers.

If $\beta>0$ (unstable stratifications), the system is still stable for all perturbations with wave numbers $\alpha>\alpha_c'$, where

$$\alpha_c'=\left[\frac{g\,\beta\,\varrho_0}{T_{s0}}\right]^{1/2}. \tag{29}$$

But for $\beta>0$ (unstable stratification), the system is unstable for all wave numbers in the range $0<\alpha<\alpha_c'$.

3. Oldroydian Viscoelastic Fluid

Here we consider the same problem as described in Sect. 2 except that the fluid is Oldroydian instead of Newtonian. The Oldroydian viscoelastic fluid is defined by the relations

$$\begin{aligned}&T_{ij}=-p\,\delta_{ij}+\tau_{ij},\\ &\left(1+\lambda\frac{\mathrm{d}}{\mathrm{d}t}\right)\tau_{ij}=2\mu\left(1+\lambda_0\frac{\mathrm{d}}{\mathrm{d}t}\right)e_{ij},\\ &e_{ij}=\frac{1}{2}\left(\frac{\partial q_i}{\partial x_j}+\frac{\partial q_j}{\partial x_i}\right).\end{aligned} \tag{30}$$

Here T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, λ and $\lambda_0\,(<\lambda)$ denote respectively the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, stress relaxation time and strain retardation time. $\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\boldsymbol{q}\cdot\nabla$ stands for the convective derivative. Relations of the type (30) were first proposed by Jeffreys for the Earth and studied by Oldroyd [2]. Oldroyd [2] also showed that many rheological equations of state, of general validity, reduce to (30) when linearized.

When a fluid flows through an isotropic and homogeneous porous medium, the gross effect is represented by Darcy's law. As a result, the resistance term replaces the usual viscous term. This fact, together with the constitutive relations (30) yields the linearized perturbed equations of motion for the Oldroyd viscoelastic fluid through a porous medium as

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\varrho}{\varepsilon}\frac{\partial \boldsymbol{q}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla\partial p-g\,\partial p\,\boldsymbol{\lambda}+\left\{T_s\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta z_s\right\}\partial(z-z_s)\,\boldsymbol{n}_s\right]-\left(1+\lambda_0\frac{\partial}{\partial t}\right)\frac{\mu}{k}\boldsymbol{q}. \tag{31}$$

Equations (5) and (6) remain unaltered. Eliminating u, v, ∂p, $\partial\varrho$ and ∂z_s from (5), (6), (13) and (31), and using (7), we obtain

$$D\left[\left\{(1+\lambda n)\frac{\varrho}{\varepsilon}n+(1+\lambda_0 n)\frac{\mu}{k}\right\}D w\right]=(1+\lambda n)\,\alpha^2\left[\frac{\varrho}{\varepsilon}n w-\frac{g(D\varrho)w}{\varepsilon n}+\frac{\alpha^2 T_s}{\varepsilon n}w_s\,\delta(z-z_s)\right]+\alpha^2(1+\lambda_0 n)\frac{\mu}{k}w. \tag{32}$$

a) Two Uniform Oldroydian Fluids Separated by a Horizontal Boundary

Here we consider the case of two uniform Oldroydian viscoelastic fluids of densities, viscosities ϱ_1, μ_1; ϱ_2, μ_2 and medium porosities, permeabilities ε_1, k_1; ε_2, k_2, separated by a horizontal boundary at $z=0$. Then for both regions of the fluid, (32) reduces to (15), and so the solutions are given by (16). The jump condition is given by

$$\Delta_0\left[\left\{(1+\lambda n)\frac{\varrho}{\varepsilon}+(1+\lambda_0 n)\frac{\mu}{n k}\right\}D w\right]=-(1+\lambda n)\frac{\alpha^2}{\varepsilon n^2}\left[g(\varrho_2-\varrho_1)-\alpha^2 T\right]w_0, \tag{33}$$

where w_0 is the value of w at $z=0$. Applying the condition (33) to the solutions (16), we obtain the dispersion relation

$$\lambda(\varrho_2+\varrho_1)\,n^3+\left[(\varrho_2+\varrho_1)+\lambda_0\left(\frac{\mu_2\varepsilon_2}{k_2}+\frac{\mu_1\varepsilon_1}{k_1}\right)\right]n^2+\left[\left(\frac{\mu_2\varepsilon_2}{k_2}+\frac{\mu_1\varepsilon_1}{k_1}\right)+\lambda\alpha\{\alpha^2 T-g(\varrho_2-\varrho_1)\}\right]+\alpha[\alpha^2 T-g(\varrho_2-\varrho_1)]=0. \tag{34}$$

It is evident from (34) that the surface tension succeeds in stabilizing all wave numbers $\alpha>\alpha_c$, where

$$\alpha_c=\left[\frac{g(\varrho_2-\varrho_1)}{T}\right]^{1/2}.$$

b) The Case of Exponentially Varying Stratifications

Using stratifications of the form (20), (32) becomes

$$\left[(1+\lambda n)\frac{\varrho_0}{\varepsilon_0}+(1+\lambda_0 n)\frac{\mu_0}{n k_0}\right](D^2-\alpha^2)\,w=\frac{(1+\lambda n)}{\varepsilon n^2}[\alpha^2 T_{s0}-g\varrho_0\beta]\,w. \tag{35}$$

Proceeding exactly as in Sect. 2(b), we obtain the dispersion relation

$$\lambda\varrho_0 n^3+\left(\varrho_0+\frac{\lambda_0\varepsilon_0\mu_0}{k_0}\right)n^2+\left[\frac{\varepsilon_0\mu_0}{k_0}+\frac{\lambda\alpha^4 d^2}{\alpha^2 d^2+s^2\pi^2}\{\alpha^2 T_{s0}-g\varrho_0\beta\}\right]n+\frac{\alpha^2 d^2}{\alpha^2 d^2+s^2\pi^2}(\alpha^2 T_{s0}-g\varrho_0\beta)=0. \tag{36}$$

It is evident from (36) that the system is stable for disturbances of all wave numbers for the stable stratification ($\beta<0$). For unstable stratification ($\beta>0$), the system is unstable for all wave numbers in the range $0<\alpha<\alpha_c'$ where

$$\alpha_c'=\left[\frac{g\beta\varrho_0}{T_{s0}}\right]^{1/2}.$$

However, for unstable stratification ($\beta>0$), the system is stable for all wave numbers $\alpha>\alpha_c'$.

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